Math 72 8.6 Equations that are Quadratic in Form

AKA Reducible to Quadratic

Solving Equations Which are Reducible to Quadratic

Objectives:

1) Solve equations using methods that change a problem into a quadratic.
   - Substitute for an expression which appears twice, u^2 and u
   - Multiply by the LCD to clear fractions
   - Square both sides (using parentheses to square the *entire* side)

Summary of Methods for solving quadratic equations

- Factor: Set =0, factor, set factors =0, isolate variable in each factor
- Square root property: Isolate the square, then square root property
- Complete the Square and square root property
- Quadratic formula
- Check by graphing (intersection or x-intercept method) – might be approximate results!

Solve.

✓ 1) $x-\sqrt{x}-6=0$ extraneous

✓ 2) $x^{2/3}-5x^{1/3}+6=0$ Substitution

✓ 3) $x^3+27=0$ factor & QF/CTS

Find the x-intercepts of:

✓ 4) $f(x)=(x-3)^2-3(x-3)-4$

Identify the strategy (or options) for solving, and write the first step(s).
(Full solutions available in class notes online.)

5) $2x=\sqrt{11x+3}$

6) $x^{-2}-x^{-1}-6=0$

✓ 7) $t^{2/5}-t^{1/5}-2=0$ 5th power

8) $p^4-3p^2-4=0$

9) $\dfrac{x}{x-1}+\dfrac{1}{x+1}=\dfrac{2}{x^2-1}$

10) $\sqrt{9x}-2=x$

11) $3+\dfrac{1}{2p+4}=\dfrac{10}{(2p+4)^2}$

✓ 12) $(5+\sqrt{r})^2+6(5+\sqrt{r})+2=0$ QF

13) $\dfrac{3x}{x-2}-\dfrac{x+1}{x}=\dfrac{6}{x-2}$

Challenge Problem

14) $x^5(x^2-25)+13x^3(x^2-25)+36x(25-x^2)=0$

Math 70

Solving Equations by Quadratic Methods

1) Summarize quadratic methods
2) Solve equations that are quadratic in form
3) Applications:
   - more work-rate problems
   - more D=RT problems
   - more geometry problems

Review:
Methods for Solving Quadratic Equations
$ax^2 + bx + c = 0$

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1) If $(dx+e)^2 = f$, use square root property.
   (already factored into a perfect square)

2) If factorable ($D = b^2 - 4ac$ is a perfect square)
   factor
   set factors = 0
   solve each sub-problem

3) Use quadratic formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

4) Complete the square, then solve by the square root property

5) Approximate solutions using GC
   method 1 intersection
   If RHS $\neq 0$.

6) Approximate solutions using GC
   method 2 x-intercept
   If RHS = 0.

Next:
Today we will solve equations which may or may not be degree 2 equations, but which can be solved using quadratic methods by observing that they are quadratic in form.

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Be cautious of:

1) If you raise both sides of equation to an even power, any of the solutions may be extraneous.

   Check by plugging into equation before you squared.

2) If you raise both sides of equation to any power, use parentheses around entire LHS and entire RHS.

   This may mean FOIL.

3) To isolate $\sqrt[3]{x}$, cube both sides.

   To isolate $\frac{1}{x}$, take reciprocal both sides.

4) You may get complex $(a+bi)$ answers. These are not extraneous and should be in your final answer.

   They cannot be checked from GC (graph).

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Solve

① $x - \sqrt{x} - 6 = 0$

Method 1: Use u-substitution

$u = \sqrt{x}$ so $u^2 = (\sqrt{x})^2 = x$

$u^2 - u - 6 = 0$

$(u-3)(u+2) = 0$ factor

$u = 3 \quad u = -2$ solve

$\sqrt{x} = 3 \quad \sqrt{x} = -2$ replace u by $\sqrt{x}$

$x = 9$ reject $x = 4$ square both sides

$\sqrt{x}$ using radical requires answer to be positive.

-OR-

plug in to find out $x=4$ is extraneous.

$4 - \sqrt{4} - 6 = 0$?

$4 - 2 - 6 = 0$

$-4 \neq 0$ no!

$\boxed{x=9}$

Method 2: Isolate the $\sqrt{\ }$ then square both sides.

$x - 6 = \sqrt{x}$

caution SQUARE entire LHS, using FOIL

$(x-6)^2 = (\sqrt{x})^2$

$x^2 - 12x + 36 = x$ FOIL, simplify

$x^2 - 13x + 36 = 0$ set $=0$

$(x-9)(x-4) = 0$ factor

$x = 9, 4$ solve factors $= 0$.

Must check in original equation, before squaring!

$9 - 6 = \sqrt{9} \qquad 4 - 6 = \sqrt{4}$

$9 - 6 = 3$ ✓ $\qquad -2 = \sqrt{4}$ no

$\boxed{x=9}$

M70

Solve

(2) $x^{2/3} - 5x^{1/3} + 6 = 0$

Solve by u-substitution

$u = x^{1/3}$

$u^2 = (x^{1/3})^2 = x^{2/3}$

$u^2 - 5u + 6 = 0$

$(u-3)(u-2) = 0$

$u = 3 \qquad u = 2$

$x^{1/3} = 3 \qquad x^{1/3} = 2$

have cube root, take cube of both sides

$(x^{1/3})^3 = x^{\frac{1}{3}\cdot\frac{3}{1}} = x^1 = x$

$x = 3^3 \qquad x = 2^3$

$\boxed{x = 27, 8}$

DO NOT DO THIS?

CAUTION! Do not attempt to isolate the cube root... and cube both sides

$\sqrt[3]{x^2} - 5\sqrt[3]{x} + 6 = 0$

$\sqrt[3]{x^2} + 6 = 5\sqrt[3]{x}$

$(\sqrt[3]{x^2} + 6)^3 = (5\sqrt[3]{x})^3$

$\underbrace{(\sqrt[3]{x^2} + 6)(\sqrt[3]{x^2} + 6)}(\sqrt[3]{x^2} + 6) = 25x$

$(\sqrt[3]{x^4} + 12\sqrt[3]{x^2} + 36)(\sqrt[3]{x^2} + 6) = 25x$

$\sqrt[3]{x^6} + 6\sqrt[3]{x^4}$
$+ 12\sqrt[3]{x^4} + 72\sqrt[3]{x^2}$
$+ 36\sqrt[3]{x^2} + 216 = 25x$

$x^2 + 18\sqrt[3]{x^4} + 108\sqrt[3]{x^2} + 216 = 25x$

ARG!!

our problem just got way worse.... ☹

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(3) Solve $x^3 + 27 = 0$

$$(x+3)(x^2 - 3x + 9) = 0$$ factor sum of cubes

$x + 3 = 0$ $\qquad$ $x^2 - 3x + 9 = 0$

$\boxed{x = -3}$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 - 36}}{2}$$

$$x = \frac{3 \pm \sqrt{-27}}{2}$$

$$\boxed{x = \frac{3}{2} \pm \frac{3i\sqrt{3}}{2}}$$

Caution:

If you isolate the cube and cube root, you will find the real solution but not the complex ones.

"Solve." means all solutions real or complex.

"Solve for real solutions" means real only.

$$x^3 = -27$$

$$\sqrt[3]{x^3} = \sqrt[3]{-27}$$

$$x = -3 \leftarrow \text{missing } \frac{3}{2} \pm \frac{3\sqrt{3}}{2} i$$

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(4) Find the x-intercepts of

$f(x) = (x-3)^2 - 3(x-3) - 4$

x-int: a point $(x, 0)$ where the graph crosses the x-axis. Find by setting $y=0$ (or replacing $f(x)$ by 0).

$0 = (x-3)^2 - 3(x-3) - 4$

<u>Method 1</u>: Solve by substitution

$u = x - 3$

rewrite $\quad 0 = u^2 - 3u - 4$

factor $\quad 0 = (u-4)(u+1)$ $\qquad \begin{array}{ccc} & -4 & \\ -4 & \times & +1 \\ & -3 & \end{array}$

option 1: replace u by x-3 inside factors

$0 = (x-3-4)(x-3+1)$

$0 = (x-7)(x-2)$

$\boxed{x=7 \quad x=2}$

option 2: solve for u, then replace u by x-3:

$u - 4 = 0 \qquad u + 1 = 0$

$u = 4 \qquad u = -1$

$x - 3 = 4 \qquad x - 3 = -1$

$\boxed{x = 7 \qquad x = 2}$

<u>Method 2</u>: Multiply and combine

$0 = (x-3)(x-3) - 3x + 9 - 4$ $\qquad$ FOIL, dist

$0 = x^2 - 6x + 9 - 3x + 5$

$0 = x^2 - 9x + 14$ $\qquad \begin{array}{ccc} & 14 & \\ -7 & \times & -2 \\ & -9 & \end{array}$ $\qquad$ combine

$(x-7)(x-2)$ $\qquad$ factor

$\boxed{x=7, \quad x=2}$

Math 70

(5) $2x = \sqrt{11x+3}$

Isolate the square root
Square both sides

$(2x)^2 = (\sqrt{11x+3})^2$

$4x^2 = 11x + 3$

$4x^2 - 11x - 3 = 0$

$(4x+1)(x-3) = 0$

$x = -\frac{1}{4} \qquad x = 3$

check for extraneous any time you raise both sides of an equation to an even power.

$2\left(-\frac{1}{4}\right) = \sqrt{11\left(-\frac{1}{4}\right)+3}$

$-\frac{1}{2} = \sqrt{\frac{1}{4}}$

$-\frac{1}{2} \neq \frac{1}{2}$ reject

principle square root
$\sqrt{\frac{1}{4}} = \frac{1}{2}$ always positive

$2(3) = \sqrt{11(3)+3}$

$6 = \sqrt{36}$

$6 = 6$ ✓

$\boxed{x = 3}$

M70

(6) Solve

$$x^{-2} - x^{-1} - 6 = 0$$

<u>Method 1:</u> u substitution

$u = x^{-1}$

$u^2 = x^{-2}$

$$u^2 - u - 6 = 0$$

$$(u-3)(u+2) = 0$$

$u = 3 \qquad u = -2$

$\frac{1}{x} = 3 \qquad \frac{1}{x} = -2$

$1 = 3x \qquad 1 = -2x$

$\boxed{\frac{1}{3} = x \qquad \frac{-1}{2} = x}$

- -

<u>Method 2:</u> write as positive exponents in denominator, mult by LCD $= x^2$

$$x^{-2} - x^{-1} - 6 = 0$$

$$\frac{1}{x^2} - \frac{1}{x} - 6 = 0$$

$$\frac{1}{x^2} \cdot x^2 - \frac{1}{x} \cdot x^2 - 6 \cdot x^2 = 0 \cdot x^2$$

$$1 - x - 6x^2 = 0$$

$$0 = 6x^2 + x - 1$$

$$0 = (3x-1)(2x+1)$$

$\boxed{x = \frac{1}{3} \qquad x = -\frac{1}{2}}$

M70

(7) $t^{2/5} - t^{1/5} - 2 = 0$

ONLY ONE VALID METHOD! Substitution

$u = t^{1/5}$ $\quad$ $u^2 = (t^{1/5})^2 = t^{2/5}$

$u^2 - u - 2 = 0$ $\quad$ rewrite in u

$(u-2)(u+1) = 0$ $\quad$ -2 -2 $+1$ -1 $\quad$ factor

option 1: replace u by $t^{1/5}$ in factors

$(t^{1/5} - 2)(t^{1/5} + 1) = 0$

$t^{1/5} - 2 = 0$ $\qquad$ $t^{1/5} + 1 = 0$

$t^{1/5} = 2$ $\qquad$ $t^{1/5} = -1.$

$(t^{1/5})^5 = 2^5$ $\qquad$ $(t^{1/5})^5 = (-1)^5$

$\boxed{t = 32 \qquad t = -1}$

option 2: Solve for u, then replace u by $t^{1/5}$.

$u = 2$ $\qquad$ $u = -1$

$t^{1/5} = 2$ $\qquad$ $t^{1/5} = -1$

$(t^{1/5})^5 = 2^5$ $\qquad$ $(t^{1/5})^5 = (-1)^5$

$\boxed{t = 32 \qquad t = -1}$

M70

(8) Solve

$p^4 - 3p^2 - 4 = 0$

$u^2 - 3u - 4 = 0$

$(u-4)(u+1) = 0$

$u = 4 \qquad u = -1$

$p^2 = 4 \qquad p^2 = -1$

$p = \pm 2 \qquad p = \pm i$

$\boxed{p = \pm 2, \pm i}$

u-substitution

$u = p^2$ so $u^2 = p^4$

just like chapter 6

M70

(9) Solve

$$\frac{x}{x-1} + \frac{1}{x+1} = \frac{2}{x^2-1}$$

domain $x \neq 1, -1$ $\qquad$ $x^2-1 = (x-1)(x+1)$

LCD $= (x-1)(x+1)$

$$\frac{x}{\cancel{(x-1)}} \cdot \cancel{(x-1)}(x+1) + \frac{1}{\cancel{(x+1)}} \cdot (x-1)\cancel{(x+1)} = \frac{2}{\cancel{(x+1)}\cancel{(x-1)}} \cdot \cancel{(x+1)}\cancel{(x-1)}$$

$x(x+1) + x-1 = 2$

$x^2 + x + x - 1 = 2$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3, 1$

reject $x=1$

$\boxed{x = -3}$

M70

(10) Solve $\sqrt{9x} - 2 = x$

$\sqrt{9x} = x + 2$ Isolate the $\sqrt{9x}$

$(\sqrt{9x})^2 = (x+2)^2$ Square both sides
* entire RHS *

$9x = x^2 + 4x + 4$

$0 = x^2 - 5x + 4$ set $= 0$

$0 = (x-4)(x-1)$ factor

$x = 4 \quad x = 1$ solve

$\sqrt{9 \cdot 4} - 2 = 4 \qquad \sqrt{9 \cdot 1} - 2 = 1$ check

$\sqrt{36} - 2 = 4 \qquad \sqrt{9} - 2 = 1$

$6 - 2 = 4 \qquad 3 - 2 = 1$

✓ ✓

$\boxed{x = 4, 1}$

M70

(11) Solve $3 + \dfrac{1}{2p+4} = \dfrac{10}{(2p+4)^2}$

domain $p \neq -2$

$3(2p+4)^2 + (2p+4) = 10$

clear fractions, mult by LCD.

$3(2p+4)(2p+4) + 2p+4 = 10$

$3(4p^2+16p+16) + 2p+4 = 10$

$12p^2 + 48p + 48 + 2p + 4 = 10$

$\dfrac{12p^2}{2} + \dfrac{50p}{2} + \dfrac{42}{2} = \dfrac{0}{2}$

divide both sides by 2

$6p^2 + 25p + 21 = 0$

$D = b^2 - 4ac$

$25^2 - 4(6)(21)$

$625 - 504 = 11^2$

it does factor!

$(6x+7)(x+3) = 0$

$\boxed{x = -\frac{7}{6} \qquad x = -3}$

M170

12 $(5+\sqrt{r})^2 + 6(5+\sqrt{r}) + 2 = 0$

Substitution

$u = 5+\sqrt{r} \Rightarrow u^2 = (5+\sqrt{r})^2$

$u^2 + 6u + 2 = 0$ does not factor!

Solve by CTS or QF

Method 1: Quadratic formula

$u = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(2)}}{2(1)}$

$u = \dfrac{-6 \pm \sqrt{28}}{2}$

$u = \dfrac{-6}{2} \pm \dfrac{2\sqrt{7}}{2}$

$u = -3 \pm \sqrt{7}$

Method 2: CTS

$u^2 + 6u + 9 = -2 + 9$

$\left\{ \begin{array}{l} \# = \frac{6}{2} = 3 \\ \#^2 = 3^2 = 9 \end{array} \right.$

$(u+3)^2 = 7$

$u + 3 = \pm\sqrt{7}$

$u = -3 \pm \sqrt{7}$

Replace u by $5+\sqrt{r}$:

$5+\sqrt{r} = -3+\sqrt{7}$ and $5+\sqrt{r} = -3-\sqrt{7}$

$-5 \qquad -5$

$\sqrt{r} = -8+\sqrt{7} \approx -.4$ $\qquad$ $\sqrt{r} = -8-\sqrt{7} \approx -5.6$

wide awake? They're both extraneous!!

$\boxed{\text{no solution}}$

Sleepy? → square both sides

$(\sqrt{r})^2 = (-8+\sqrt{7})^2$ $\qquad$ $(\sqrt{r})^2 = (-8-\sqrt{7})^2$

$r = (-8+\sqrt{7})(-8+\sqrt{7})$ $\qquad$ $r = (-8-\sqrt{7})(-8-\sqrt{7})$

$r = 64 - 16\sqrt{7} + 7$ FOIL $r = 64 + 16\sqrt{7} + 7$

$r = 71 - 16\sqrt{7}$ $\qquad$ $r = 71 + 16\sqrt{7}$

check for extraneous

$(5+\sqrt{71-16\sqrt{7}})^2 + 6(5+\sqrt{7-16\sqrt{7}}) + 2 =$ imaginary result $\neq 0$

$(5+\sqrt{71+16\sqrt{7}})^2 + 6(5+\sqrt{7+16\sqrt{7}}) + 2 \approx 318.9 \neq 0$

$\boxed{\text{no solution}}$

Math 70 S/e 8.3

Solve

(13) $\frac{3x}{x-2} - \frac{x+1}{x} = \frac{6}{x-2}$

Just a rational equation like chapter 6.

domain: $x \neq 0, 2$

LCD $= x(x-2)$

$\frac{3x}{x-2} \cdot x(x-2) - \frac{(x+1)}{x} \cdot x(x-2) = \frac{6}{x-2} \cdot x(x-2)$

$3x \cdot x - (x+1)(x-2) = 6x$

CAREFUL! FOIL first, then distribute negative (order of operations: mult before subtract)

$3x^2 - [x^2 - x - 2] = 6x$

$3x^2 - x^2 + x + 2 = 6x$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$x = \frac{1}{2}$ $\quad$ $x = 2$ reject

$\boxed{x = \frac{1}{2}}$

M70

(14) $$\overbrace{x^5(x^2-25)}^{\text{1st term}} + \overbrace{13x^3(x^2-25)}^{\text{2nd term}} + \overbrace{36x(25-x^2)}^{\text{3rd term}} = 0$$

(x^2-25) factor out -1 from 3rd term

$$x^5(x^2-25) + 13x^3(x^2-25) - 36x(x^2-25) = 0$$

note GCF $x\,(x^2-25)$

$$x(x^2-25)\left[x^4 + 13x^2 - 36\right] = 0$$

substitute $u = x^2$

$$x(x^2-25)(u^2+13u-36) = 0$$

$$x(x^2-25)(u-9)(u-4) = 0$$

-36

$-9 \quad -4$

$+13$

replace $u = x^2$

$$x(x^2-25)(x^2-9)(x^2-4) = 0.$$

factor three differences of squares

$$x(x+5)(x-5)(x-3)(x+3)(x+2)(x-2) = 0.$$

set factors $= 0$

$$\boxed{x = 0, -5, +5, 3, -3, -2, 2}$$